Mathematics of the *Almagest*

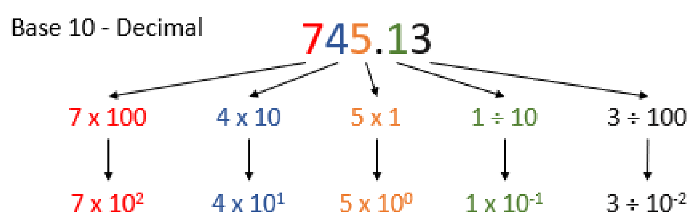
Purpose of Class

The *Almagest* is a series of thirteen books that lays out the geocentric model of the universe, written in the 2nd century by Claudius Ptolemy. At that time, much of the math that we learn in high school (in particular algebra and trigonometry[1]) had not been developed yet. As such, Ptolemy relies heavily on geometry and mathematical concepts that are often only lightly taught, if at all, as they have been replaced by modern methods which are significantly faster. As such, to be able to teach classes on Ptolemy's mathematical models, we will first need a course in his mathematical techniques.

Understanding Sexagesimal

Thinking First About Decimal

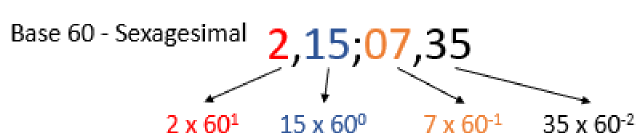
All numbering systems are rooted in a pattern. The numbering system we use everyday is "decimal," which is also known as "base 10." If we break apart a number like 745.13, we can see why:



Here, each digit in the number is in a place corresponding to a power of 10. That number is multiplied by that power and when put back together, forms the number. Each place to the left is another power of 10 higher. However, a numbering system can be formed using any number as a base. Computers use a system with 2 as a base known as "binary." Because astronomers frequently work with circles and circles are defined to have 360°, Ptolemy used a base that had a lot of common factors with that – He used base 60, known as "sexagesimal." In fact, the world is full of non-base 10 systems we use every day: time, rulers, etc…

A Sexagesimal Number Follows the Same Pattern

Let's take an example of a sexagesimal number:



Here, we can see the same pattern: numbers are coefficients of a series of powers of 60. While Ptolemy didn't use Arabic numerals as displayed here[2], this is how numbers look in modern translations for ease of use. In addition, translators insert commas between the places and a semi-colon to denote where the power switches to negative.

Converting Sexagesimal to Decimal

If we want to convert from sexagesimal to decimal we can do so by applying the definition of the powers we just discussed:

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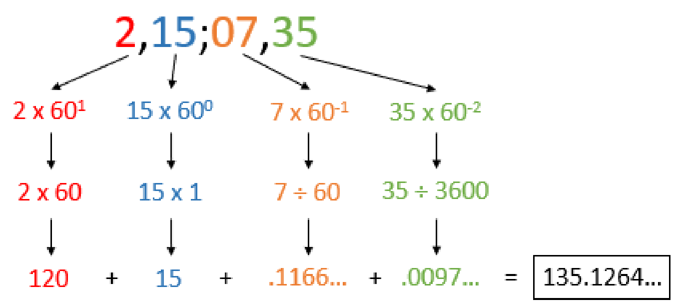
[1] Some of the concepts in these fields did exist, but weren't formalized yet

[2] - Ptolemy used a set of alphanumeric symbols that were actually base 10 but slotted into the various places for the sexagesimal. In this system, $\alpha = 1$, $\beta = 2$, etc… Different Greek letters were used to denote the 10's and 100's places.



Practice Problems

Convert the following sexagesimal numbers to decimal.

1. 3,16;29,09
2. 1,18;55,53
3. 1,03;00,15

Math Still Works with Different Bases

Even though numbers expressed in sexagesimal may look funny, math still works the same. Here's a quick example to prove it. To start, we'll use the number we just converted, and then add 1,50;50,30 to it. We'll also add them in decimal form at the same time so we'll need to convert the number we're adding as well:

Sexagesimal		Decimal
2,15;07,35		135.1264...
+ 1,50;50,30	1 x 60 + 50 x 1 + 50 ÷ 60 + 30 ÷ 3600 =	+ 110.8417...

Now we can add each of these together place by place:

Sexagesimal		Decimal
2,15;07,35		135.1264...
+ 1,50;50,30	1 x 60 + 50 x 1 + 50 ÷ 60 + 30 ÷ 3600 =	+ 110.8417...
3,65;57,65		245.9681...

Looking at the sexagesimal side, we notice that some of the places are higher than 59, which means we'll need to do some carrying. To do so, we subtract 60 from any place that's over 59, and add 1 to the place to the left, and leave the remainder in the place you subtracted from.

Sexagesimal		Decimal
2,15;07,35		135.1264...
+ 1,50;50,30	1 x 60 + 50 x 1 + 50 ÷ 60 + 30 ÷ 3600 =	+ 110.8417...
3,65;57,65		245.9681...
4,05;58,05		

Now we have a final answer in sexagesimal. So let's convert and ensure that it matches our result from adding in decimal:

Sexagesimal		Decimal
2,15;07,35		135.1264…
+ 1,50;50,30	1 x 60 + 50 x 1 + 50 ÷ 60 + 30 ÷ 3600 =	+ 110.8417…
3,65;57,65		245.9681…
4,05;58,05	4 x 60 + 5 x 1 + 58 ÷ 60 + 5 ÷ 3600 =	245.9681…

Here we can see the two match, confirming that the operation did work the same in both number systems. We will see an example of subtraction later and multiplication and division become annoyingly difficult, so I generally convert to decimal when needing to perform those operations.

Converting Decimal to Sexagesimal

Since I don't generally want to do math by hand and do want to use a calculator, I frequently convert the sexagesimal to decimal as we did above, and then convert back when I'm finished. As such, I'll explore how that conversion is done using the final answer we got above in decimal: 245.9681…. That way, we know what the answer should be and can check it.

First, I consider what place I need to start in. The whole number portion is over 60^1 but not as high as 60^2 (3,600)[3]. As such, we'll be starting in the 60^1 place. So, divide the number by 60^1.

Work	Solution
245.9681 ÷ 60 ≈ 4.0995	4,XX;XX,XX

The whole number from that we'll put in the 60^1 place. The remainder, we'll multiply by 60 and repeat this process with until we reach the desired accuracy:

Work	Solution
245.9681 ÷ 60 ≈ 4.0995	4,XX;XX,XX
.0995 x 60 ≈ 5.9681	4,05;XX,XX
.9681 x 60 ≈ 58.0860	4,05;58,XX
.0860 x 60 ≈ 5.1600	4,05;58,05

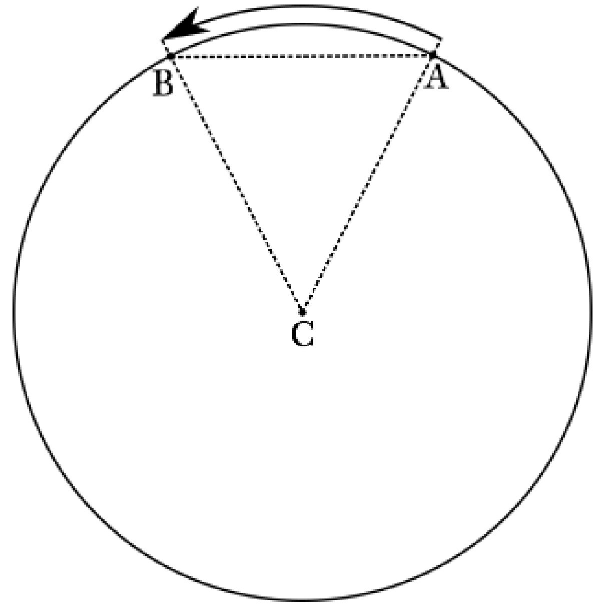
Practice Problems

Convert the following decimal numbers to sexagesimal.

1. 200.31222…
2. 351.4375
3. 93.856388…

[3] In truth, Ptolemy rarely goes above 360° since that's the maximum degrees in a circle. As such, there's almost never a case to consider the 60^2 place or higher.

Triangles

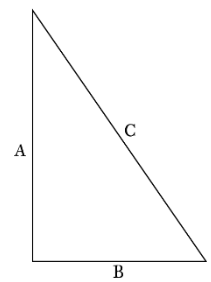


When thinking about astronomical models, triangles might not immediately come to mind. After all, the models are planets and other astronomical bodies moving in *circles*. So how to triangles come into play? First, consider an object orbiting the observer at the center of a circle, C. At some point in time, it's at point A. Then sometime later is has moved to point B. This naturally creates an angle between these two points as seen by the observer. Then, if you connect those two points, it forms a triangle.

Notation

- Sides
  - Written as the two endpoints of the side, with a line over it. (Ex: $\overline{AB}$)
  - Measured in arbitrary units called "parts" denoted by a superscript "p" (Ex: 60^p)
- Angles
  - Written as the points of the triangle with the vertex as the middle letter after a < sign (Ex: <BCA)
  - Measured in degrees (°)

The Pythagorean Theorem



For right triangles, there is a relationship between the sides and the hypotenuse:

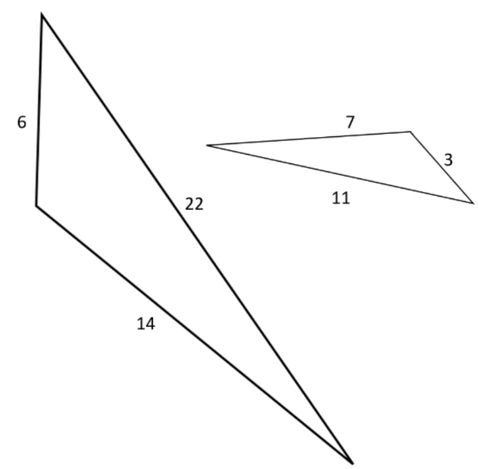
$$A^2 + B^2 = C^2$$

Here, A and B are the sides and C is the hypotenuse.

Practice Problems

1.

Similar Triangles



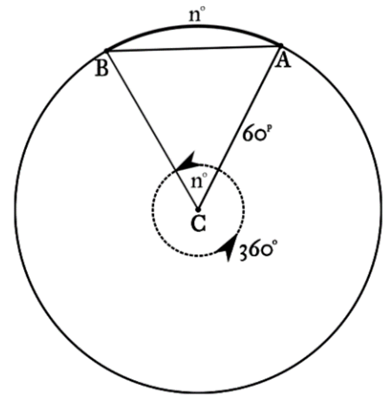
Triangles that have the same shape but are different sizes are known as similar triangles. These are useful because the corresponding sides in a triangle are always multiplied/divided by a consistent factor. For example, these two triangles are similar – just rotated. Each side of the larger triangle is 2x the length of the smaller one. Thus, if we knew all the sides of one triangle, but only knew one for the other, we could take the ratio of the sides we did know and apply that ratio to the other sides to solve for the unknown sides.

There are a few tests we can apply to determine whether or not triangles are similar. This description is the modern formalization of this, but Ptolemy makes use of these concepts even if he doesn't use these terms. Most frequently, Ptolemy knows that two triangles are the "same shape" because two triangles have all the same angles. In modern math, we would call this Angle-Angle-Angle similarity (AAA). However, other tests exist as well.

Circles

As noted above, circles feature prominently as orbits are a big part of astronomical models.

Definitions



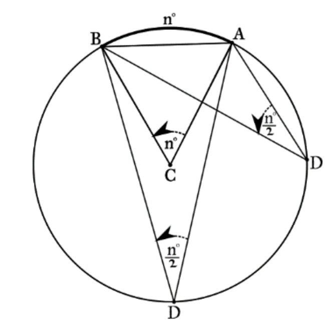
- A circle has 360°
    - Each 1° is divided into 60 minutes, denoted by a single tick mark (‘)
    - Each 1’ is divided into 60 seconds, denoted by a double tick mark (“)
- A circle has a radius of 60^p unless otherwise noted (since Ptolemy doesn’t start out knowing the radius of an orbit)
- A chord is a line segment between two points on the perimeter of a circle
- An arc is a segment of the perimeter of a circle between two points
    - Ptolemy doesn’t measure the perimeter of a circle in units of length, but instead does it in degrees. Thus, the arc always has the same measure as the central angle which it subtends.

Relationship Between Chords and Angles

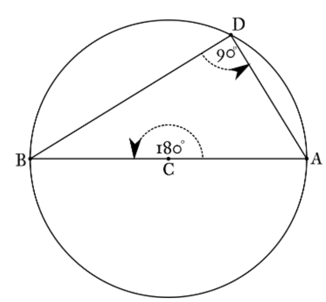
If we consider a central angle in a circle, it will intersect the circle at two points, creating a chord. The length of that chord directly corresponds to the angle. For example, if the central angle is 180°, then the chord is the diameter of the circle or 120^p. If the central angle is 90°, then it is a right triangle with sides of 60^p and we can use the Pythagorean theorem to determine the length of the chord to be $84;51,10^p$.

Ptolemy figured out the length of the chords for each ½° increment in the first book of the *Almagest* **for a circle with a radius of 60^p**. This is known as the table of chords. A copy is attached later in this handout.

Inscribed or Central Angle Theorem



The measure of an angle formed by two points on the perimeter of the circle with a vertex elsewhere on the perimeter is always half that of an angle formed by those same two points with its vertex at the center of the circle. It doesn’t matter where on the perimeter of the circle the vertex is so long as it isn’t *too close* to the points themselves. We could get into what “too close” means, but since Ptolemy only ever uses this when it’s opposite the two points, it’s not important for this class.

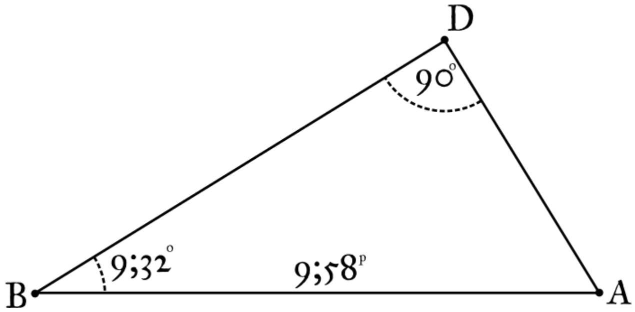


This theorem does have an interesting consequence Ptolemy takes advantage of. Specifically, what happens if the vertex on the perimeter is 90°? In that case, the central angle would be twice that or 180° and as we saw previously, that means the chord would have a measure of 120^p making it the diameter of the circle. Thus, if you have a right triangle inscribed in a circle, the hypotenuse is always the diameter of that circle.

Solving Triangles Using the "Demi Degrees" Method

Getting Started

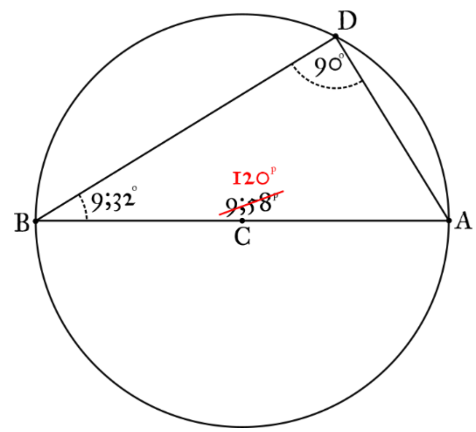
Everything we've covered so far is all to help us understand one thing: How Ptolemy solves triangles without the tools of trigonometry. Ptolemy is very good about creating right triangles, so that's what we'll focus on in this example:



Givens:

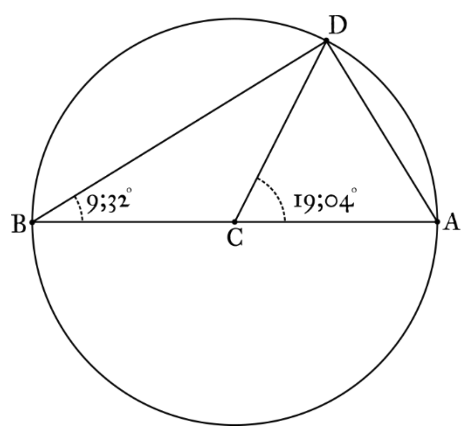
- Right triangle
- <DBA = 9;32°
- $\overline{BA} = 9;58^p$

Solve for the unknown sides of the triangle



To begin, Ptolemy draws a circle around this triangle such that the three points are all on the perimeter. Due to the special case of the Inscribed Angle Theorem, this means that $\overline{BA}$ is the hypotenuse and has a point, C, on it which is at the center of the circle. However, in Ptolemy's circles, we consider the diameter to be 120^p. As such, we'll change the measure of $\overline{BA}$ to that. Since this doesn't change the measure of any of the angles, what this really does is create a *different* triangle that is *similar* to the one we started with which we will solve first. To help keep track of which is which, I've colored the lengths of the sides where we're working in that similar triangle in red.

Next, we'll draw in a line from the center to point D. Because of the Inscribed Angle Theorem, we can state that <DCA = 19;04° since it has the same endpoints (D and A) as <DBA which is on the perimeter and therefore must have twice its measure.



Next, since we created this new triangle in a circle with a radius of 60^p, we can use the Ptolemy's table of chords to look up the length of the chord $\overline{DA}$ since we know its central angle[4]. However, we immediately run into a problem since the table is in half degree increments and 19;04° lies between two entries. As such, we'll have to estimate between the two entries which is known as interpolation. There are various ways we can do this, but I'll show the one that I think is the most intuitive.

[4] This method is referred to as using "demi-degrees". The reason is not obvious as I'm actually deviating from Ptolemy's method to explain what's going on behind the scenes – We're finding the central angle. Ptolemy doesn't explicitly do this. Instead, he uses a rhetorical trick in which he instead considers the circle we've just created about the similar triangle to be one in which "two right angles are 180°". This really doesn't make mathematical sense, but what it would mean is that the angle on the perimeter of the circle, <DBA, would double, having the same measure as the

Interpolation Interlude

Before we dive into the math, let's consider what we're trying to do. The value we want to find the corresponding chord for is 19;04°. That is 4 sixt*ieths* ($4 \cdot 60^{-1}$) past 19;00°. As such, our goal will be to find the additional chord length we need to add on to the chord length corresponding to 19;00° for each 1 sixtieth, and then do that four times.

To do so, let's look at the chord lengths corresponding to 19;00° and 19;30°. They are 19;48,21^p and 20;19,19^p respectively for a change of 0;30° or 30 minutes (30'). So we'll start by finding the difference in the chord lengths between these two. Doing so will involve some borrowing as we cannot take 21 from 19 in the 60^{-2} place, nor 48 from 19 in the 60^{-1} place.

78
19 ~~18~~ 79
~~20~~;~~19~~,~~19~~^{p}
\- 19;48,21^p
0;30,58^p

After doing the appropriate borrowing, we determine that the length of the chord increases by 0;30,58^p. Note that this is *not* a constant number as the chord length increases more rapidly for small central angles.

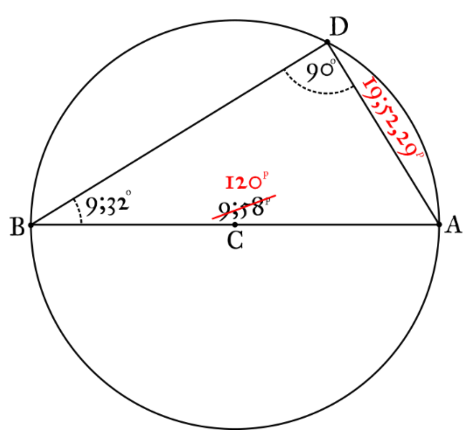
Again, this was for a difference of 30', and we want to know for each 1', we we'll need to divide that result by 30'. I convert this to decimal, do the division, and then convert back. Doing so, I get the increment for each minute of central angle to be 0;01,01,56°. If you look at Ptolemy's table of chords, you'll see a column called "Sixtieths" that did this math for us. However, I'm showing how to do this sort of interpolation here because in the other tables Ptolemy creates, he does not have such a column. Still, in this table, he came up with 0;01,01,57° which is a slight difference in rounding from what we got.

But this was for 1 minute. We have 4. So, we need to multiply that by 4 which gives us 0;04,08^p. But again, this is *in addition to* the chord we had at 19;00° so we need to add it on:

19;41,28^p + 0;04,08^p = **19;52,29^p**

Thus, the length of the chord corresponding to a central angle of 19;04° is 19;52,29^p for a circle where the diameter is 60^p.

Solving the Triangle



Now that we know the length of the side, $\overline{DA}$, in addition to the hypotenuse in the context of this similar triangle, we can use the Pythagorean theorem to solve for the 3rd side.

$$\overline{BD}^2 + 19;52{,}29^2 = 120^2$$

Again, as this involves multiplication, I generally convert to decimal and use a calculator to solve. Doing so and converting back, I get

$$\overline{BD} = 118;20{,}34^p$$

However, we are still in the context of the similar triangle we created. To convert back to our original triangle, we'll need to take advantage of the ratio of the side we knew in common in both: the hypotenuse. That ratio is:

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central angle. Then we could just use *that* angle to look up in the chord table. To help keep these straight, modern translators denote when we're in this alternate context by a double or demi-degrees symbol (°°) hence where the name comes from. To me, this odd convention serves two purposes for Ptolemy: 1) It helps keep track of which context we're in (the regular triangle or the similar one we constructed) and 2) Saves the step of having to explicitly define the central angle. Regardless, I personally find it more confusing than helpful and tend to ignore it, thinking about this method in relation to the central angle as shown here.

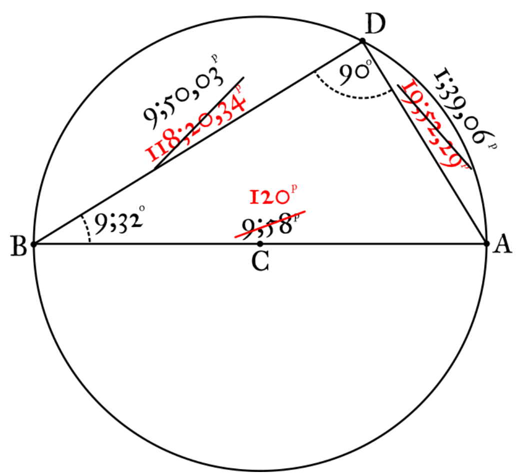
$$\frac{9;58^p}{120^p} = 0.0831$$

If we apply this same ratio to the other sides we found, we will have solved the triangle.

$$\overline{BD} = 118;20,34^p \cdot 0.0831 \ldots = 9;50,03^p$$

$$\overline{DA} = 19;52,29^p \cdot 0.0831 \ldots = 1;39,06^p$$

Thus, our final triangle looks like this:

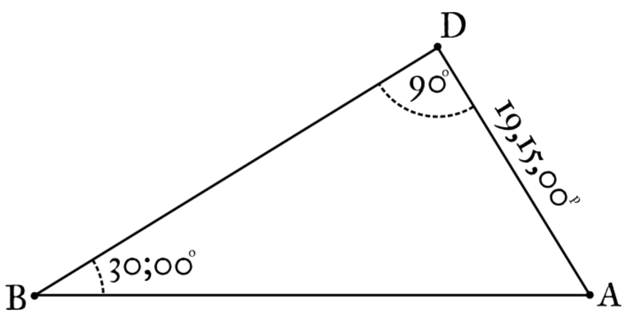


Note that it doesn't matter which side of the triangle we were originally given so long as we correctly assign it to the corresponding side to get the ratio for the scale of the similar triangles.

Practice Problems

Solve for the unknown sides

1.



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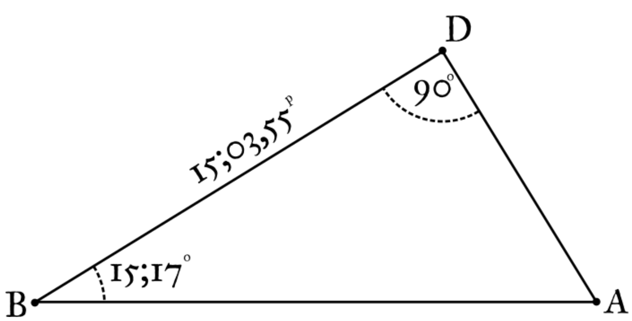
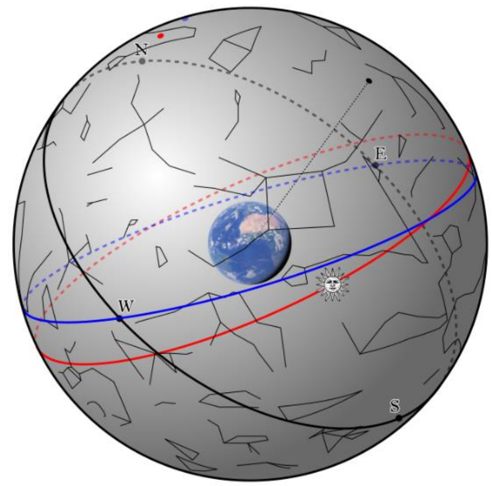


Speaking Spherically

In my previous class, *De Sphaera (On the Sphere)*, we began exploring the celestial sphere, defining astronomical terms, coordinate systems, and how to visualize such things. The most important concept from that class as it pertains to this class is the idea of a “great circle” which is any circle drawn around a sphere which has a center that is also the center of the sphere. Thinking about the earth, the equator is a great circle, but any other line of latitude is not since it would have a center that is not the center of the earth.

The concept of great circles comes up frequently in astronomy because it *looks* like we are the center of the cosmos. Thus, things often trace great circles as the celestial sphere spins around us[5].



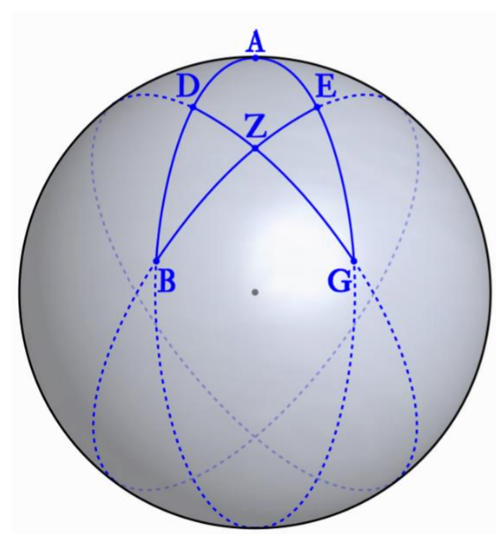
For example, consider the celestial equator (earth’s equator just extended outwards to the celestial sphere, which is drawn in blue here) and the path the sun seems to make every year (known as the ecliptic, in red). These are both great circles but are tilted with respect to one another. So too is the apparent horizon for an observer a great circle (in black).

The intersection of these great circles can often create triangles that Ptolemy may want to solve. For example, what is the angular distance between the ecliptic and celestial equator at a certain ecliptic longitude? What is the altitude of a star above an observer’s horizon at a given time?

While you may be tempted to try to apply some of the math we just covered, it turns out we can’t. That type of math only works on flat surfaces and a sphere is anything but. We can take the plane of a great circle and do math within it, but as soon as we move out of that plane, we’re in an entirely separate field of math known as spherical geometry meant for solving such problems. Fortunately, Ptolemy seems to use only *one* theorem exclusively to solve them.

Menelaus’ Theorem

There are several variations on Menelaus’ theorem, most of them meant for flat surfaces. But the one Ptolemy uses is a spherical version which he derives in Book I Chapter 13 of the *Almagest*. This theorem involves the intersection of four great circles. So long as the circles do not have more than two circles intersecting at a single point, there will always be shape created looking something like the one pictured here[6].



What Menelaus’ theorem does is allows us to relate the chords corresponding to the various arcs to one another taking a spherical geometry problem and breaking it down into what may be several flat ones. There are two versions of the theorem. Each one involves the chords of six of the arcs. Thus, we will need to have the other five as knowns to be able to solve for the sixth. The first version is:

$$\frac{Crd\ arc\ 2GE}{Crd\ arc\ 2EA} = \frac{Crd\ arc\ 2GZ}{Crd\ arc\ 2ZD} \cdot \frac{Crd\ arc\ 2DB}{Crd\ arc\ 2BA}$$

The second version is:

$$\frac{Crd\ arc\ 2GA}{Crd\ arc\ 2EA} = \frac{Crd\ arc\ 2GD}{Crd\ arc\ 2DZ} \cdot \frac{Crd\ arc\ 2ZB}{Crd\ arc\ 2BE}$$

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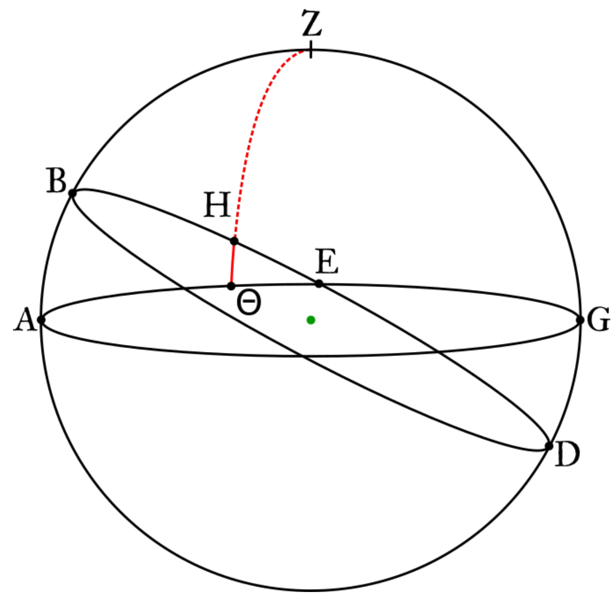
[5] Ptolemy envisioned the entire cosmos spinning around us once every day and the earth did not rotate. Later astronomers held that the earth did the turning.

[6] I always envision it as looking a bit like a Star Trek badge or a 5-pointed star missing the points on the left and right.

Both versions of the equation work the same, but just have different chords allowing for some flexibility in what may be solved for. Take some time to look at what pieces of the shape are being used for each form. The first one uses each half of the outer side for one side, a half and the whole of the other, and then each piece of one of the diagonals. The second uses a whole side and a part for both outer sides, as the whole diagonal and a part.

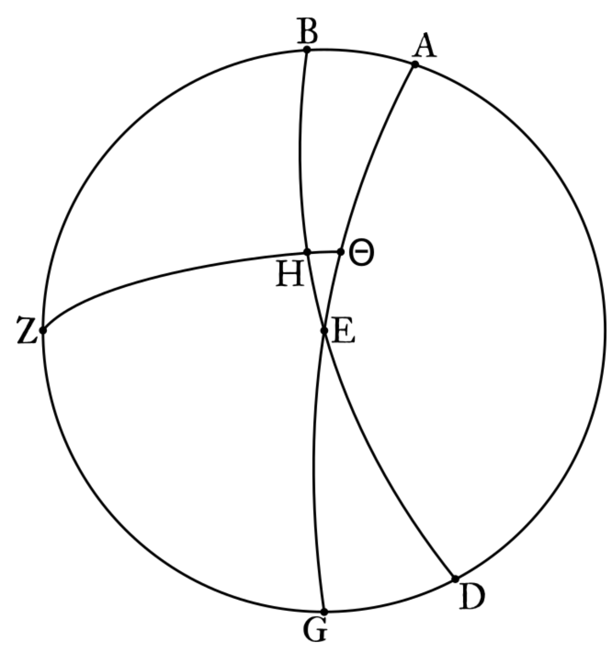
While these formulae look complex, what each term is really saying is that we need to find the chord corresponding to twice the respective arc. For example, if *arc GE* was 20°, then we would look up the chord for twice that or 40° which is 41;02,33^p.

Example Problem



To illustrate, let's walk through the first time Ptolemy actually uses this theorem in the *Almagest*. There, Ptolemy wants to find the length of an arc (measured in degrees) between the ecliptic (circle BED) and celestial equator (circle AEG), shown here as $arc\ H\Theta$ which is part of a great circle going through the north celestial pole at Z. That's three great circles, so what's the fourth?

It's the one that looks like the perimeter of this circle which goes through points A, B, G, and D. Points B and D are important because they are the furthest the sun gets away from the celestial equator which are the equinoxes. From measurement, we know that the most they get away is 23;30°[7], thus $arc\ AB$ is 23;30°. Meanwhile, the point where the two great circles intersect, such as point E, are the equinoxes and they are 90° away from the solstice, so we know both $arc\ BE$ and arc AE are 90°. Since Z is the north celestial pole, it is, by definition, 90° away from the celestial equator so $arc\ Z\Theta$ is 90° (as is the arc from Z to any other point on the celestial equator).



Similar to the chord table, Ptolemy plans to create a table with the length of this arc measured every 1° around the ecliptic, so we can pick what value we'd like to use for $arc\ EH$. For this example, we'll chose 30°.

That gives us a bunch of arc lengths to play with, so let's work out how to apply Menelaus' theorem. First, we need to find the shape so we can figure out what the corresponding pieces are. It's a bit easier if I redraw the image with less detail and rotated a bit. Doing so, we can see the shape, rather distorted towards the top of this image. It especially helps if we ignore points G and D.

However, we must figure out which form of the equation we can use. Looking at the above, we stated we know the following:

- $arc\ ZA\ =\ 90°$
- $arc\ AB$ = 23;30°
- $arc\ \Theta Z\ =\ 90°$
- $\text{arc HE}\ =\ 30°$
- $\text{arc EB}\ =\ 90°$

[7] This is the angle between the ecliptic and celestial equator. I have presented it here as 23;30° for simplicity but Ptolemy uses a value of 23;51° which is high.

Looking at which arcs these are, we can see that we have the outer legs of the shape, plus the upper halves of those legs which means we should use the second version of Menelaus' theorem. Translating all the points we can write:

$$\frac{Crd\ arc\ 2ZA}{Crd\ arc\ 2AB} = \frac{Crd\ arc\ 2\Theta Z}{Crd\ arc\ 2\Theta H} \times \frac{Crd\ arc\ 2HE}{Crd\ arc\ 2EB}$$

In this equation, we know all the pieces except the one corresponding to the arc which we wanted to solve for. Thus, we can start putting in the pieces, recalling that we need to look up the chord corresponding to *twice* the length of the arc. Fortunately, many of these are the same which makes things a bit easier:

- $Crd\ arc\ 2(90°) = 120^p$
- $Crd\ arc\ 2(23;30°) = 47;51{,}00^p$
- $Crd\ arc\ 2(30°) = 60;00{,}00^p$

Plugging into our equation:

$$\frac{120^p}{47;51{,}00^p} = \frac{120^p}{Crd\ arc\ 2\Theta H} \times \frac{60^p}{120^p}$$

Solving the equation, we get:

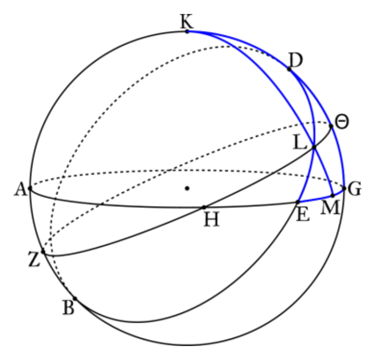
$$Crd\ arc\ 2\Theta H = 23;55{,}30^p$$

To remind ourselves, this is the length of the chord corresponding to *twice* $arc\ \Theta H$. Thus, to complete this process, we need to find what arc has a chord with that length and then divide by two. Looking at the table of chords, we find this is *very* close to an arc of 23;00°. We could do some interpolation to determine this a bit more accurately, but for the purposes of this example, we can call it good enough. Half of that arc would be 11;30°. Thus, $arc\ \Theta H\ \approx 11;30°$.

Practice Problems

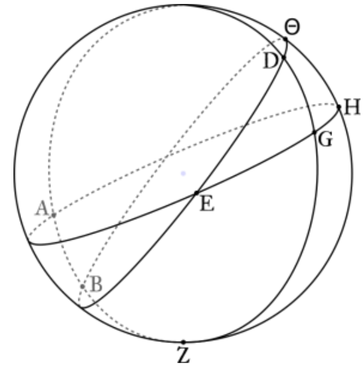
1. In the image to the right, the following arcs are known:
   - $arc\ DK\ =\ 36;00°$
   - $arc\ DG\ =\ 54;00°$
   - $arc\ LM\ =\ 11;39{,}59°$
   - $arc\ KL\ =\ 72;20{,}01°$
   - $arc\ EG\ =\ 90;00°$

   Determine which form of Menelaus' equation to use and solve for $arc\ EM$. The Menelaus configuration is highlighted in blue.



2. In the image to the right, the following arcs are known:
   - $arc\ EH\ =\ arc\ E\Theta =\ arc\ DZ\ =\ 90°$
   - $arc\ EG\ =\ 77;41°$
   - $arc\ DG\ =\ 31;20°$

   Determine which form of Menelaus' equation to use and solve for $arc\ \Theta H$.



Ptolemy's Table of Chords

Arc	Chord	Sixtieths	Arc	Chord	Sixtieths
0;30	0;31;25	0;01,02,50	**23;00**	23;55,27	0;01,01,33
1;00	1;02,50	0;01,02,50	**23;30**	24;26,13	0;01,01,30
1;30	1;34,15	0;01,02,50	**24;00**	24;56,58	0;01,01,26
2;00	2;05,40	0;01,02,50	**24;30**	25;27,41	0;01,01,22
2;30	2;37,04	0;01,02,48	**25;00**	25;58,22	0;01,01,19
3;00	3;08,28	0;01,02,48	**25;30**	26;29,01	0;01,01,15
3;30	3;38,52	0;01,02,48	**26;00**	26;59,38	0;01,01,11
4;00	4;11,16	0;01,02,47	**26;30**	27;30,14	0;01,01,08
4;30	4;42,40	0;01,02,47	**27;00**	28;00,48	0;01,01,04
5;00	5;14,04	0;01,02,46	**27;30**	28;31,20	0;01,01,00
5;30	5;45,27	0;01,02,45	**28;00**	29;01,50	0;01,00,56
6;00	6;16,49	0;01,02,44	**28;30**	29;32,18	0;01,00,52
6;30	6;48,11	0;01,02,43	**29;00**	30;02,44	0;01,00,48
7;00	7;19,33	0;01,02,42	**29;30**	30;33,08	0;01,00,44
7;30	7;50,54	0;01,02,41	**30;00**	31;03,30	0;01,00,40
8;00	8;22,15	0;01,02,40	**30;30**	31;33,50	0;01,00,35
8;30	8;53,35	0;01,02,39	**31;00**	32;04,08	0;01,00,31
9;00	9;24,54	0;01,02,38	**31;30**	32;34,22	0;01,00,27
9;30	9;56,13	0;01,02,37	**32;00**	33;04,35	0;01,00,22
10;00	10;27,32	0;01,02,35	**32;30**	33;34,46	0;01,00,17
10;30	10;58,49	0;01,02,33	**33;00**	34;04,55	0;01,00,12
11;00	11;30,05	0;01,02,32	**33;30**	34;35,01	0;01,00,08
11;30	12;01,21	0;01,02,30	**34;00**	35;05,05	0;01,00,03
12;00	12;32,36	0;01,02,28	**34;30**	35;35,06	0;00,59,57
12;30	13;03,50	0;01,02,27	**35;00**	36;05,05	0;00,59,52
13;00	13;35,04	0;01,02,25	**35;30**	36;35,01	0;00,59,48
13;30	14;06,16	0;01,02,23	**36;00**	37;04,55	0;00,59,43
14;00	14;37,27	0;01,02,21	**36;30**	37;34,47	0;00,59,38
14;30	15;08,38	0;01,02,19	**37;00**	38;04,36	0;00,59,32
15;00	15;39,47	0;01,02,17	**37;30**	38;34,22	0;00,59,27
15;30	16;10,56	0;01,02,15	**38;00**	39;04,05	0;00,59,22
16;00	16;42,03	0;01,02,13	**38;30**	39;33,46	0;00,59,16
16;30	17;13,09	0;01,02,10	**39;00**	40;03,25	0;00,59,11
17;00	17;44,14	0;01,02,07	**39;30**	40;33,00	0;00,59,05
17;30	18;15,17	0;01,02,05	**40;00**	41;02,33	0;00,59,00
18;00	18;46,19	0;01,02,02	**40;30**	41;02,33	0;00,58,54
18;30	19;17,21	0;01,02,00	**41;00**	42;01,30	0;00,58,48
19;00	19;48,21	0;01,01,57	**41;30**	42;30,54	0;00,58,42
19;30	20;19,19	0;01,01,54	**42;00**	43;00,15	0;00,58,06
20;00	20;50,16	0;01,01,51	**42;30**	43;29,33	0;00,58,31
20;30	21;21,11	0;01,01,48	**43;00**	43;58,49	0;00,58,25
21;00	21;52,06	0;01,01,45	**43;30**	44;28,01	0;00,58,18
21;30	22;22,58	0;01,01,42	**44;00**	44;57,10	0;00,58,12
22;00	22;53,49	0;01,01,39	**44;30**	44;26,16	0;00,58,06
22;30	23;24,39	0;01,01,36	**45;00**	45;55,19	0;00,58,00

Arc	Chord	Sixtieths	Arc	Chord	Sixtieths
45;30	46;24.19	0;00,57,54	**68;00**	67;06,12	0;00,52,01
46;00	46;53,16	0;00,57,47	**68;30**	67;62,12	0;00,51,52
46;30	47;22,09	0;00,57,41	**69;00**	67;58,08	0;00,51,43
47;00	47;51,00	0;00,57,34	**69;30**	68;23,59	0;00,51,33
47;30	48;19,47	0;00,57,27	**70;00**	68;49,45	0;00,51,23
48;00	48;48,30	0;00,57,21	**70;30**	69;15,27	0;00,51,14
48;30	49;17,11	0;00,57,14	**71;00**	69;41,04	0;00,51,04
49;00	48;48,60	0;00,57,7	**71;30**	70;06,36	0;00,50,55
49;30	50;14,21	0;00,57,0	**72;00**	70;32,03	0;00,50,45
50;00	50;42,51	0;00,56,53	**72;30**	70;57,26	0;00,50,35
50;30	51;11,18	0;00,56,46	**73;00**	71;22,44	0;00,50,26
51;00	51;39,42	0;00,56,39	**73;30**	71;47,56	0;00,50,16
51;30	52;08,00	0;00,56,32	**74;00**	72;13,04	0;00,50,06
52;00	52;36,16	0;00,56,25	**74;30**	72;38,07	0;00,49,56
52;30	53;04,29	0;00,56,18	**75;00**	73;03,05	0;00,49,46
53;00	53;32,38	0;00,56,10	**75;30**	73;27,58	0;00,49,36
53;30	54;00,43	0;00,56,3	**76;00**	73;52,46	0;00,49,26
54;00	54;28,44	0;00,55,55	**76;30**	74;17,29	0;00,49,16
54;30	54;56,42	0;00,55,48	**77;00**	74;42,07	0;00,49,06
55;00	55;24,36	0;00,55,40	**77;30**	75;06,39	0;00,48,55
55;30	55;52,26	0;00,55,33	**78;00**	75;31,07	0;00,48,45
56;00	56;20,12	0;00,55,25	**78;30**	75;55,29	0;00,48,34
56;30	56;47,54	0;00,55,17	**79;00**	76;19,46	0;00,48,24
57;00	57;15,33	0;00,55,9	**79;30**	76;43,58	0;00,48,13
57;30	57;43,07	0;00,55,1	**80;00**	77;08,05	0;00,48,03
58;00	58;10,38	0;00,54,53	**80;30**	77;32,06	0;00,47,52
58;30	58;38,05	0;00,54,45	**81;00**	77;56,02	0;00,47,41
59;00	59;05,27	0;00,54,37	**81;30**	78;19,52	0;00,47,31
59;30	59;32,45	0;00,54,29	**82;00**	78;43,38	0;00,47,20
60;00	60;00,00	0;00,54,21	**82;30**	79;07,18	0;00,47,09
60;30	60;27,11	0;00,54,12	**83;00**	79;30,52	0;00,46,58
61;00	60;54,17	0;00,54,4	**83;30**	79;54,21	0;00,46,47
61;30	60;21,19	0;00,53,56	**84;00**	80;17,45	0;00,46,36
62;00	61;48,17	0;00,53,47	**84;30**	80;41,03	0;00,46,25
62;30	62;15,10	0;00,53,39	**85;00**	81;04,15	0;00,46,14
63;00	62;42,00	0;00,53,30	**85;30**	81;27,22	0;00,46,03
63;30	63;08,45	0;00,53,22	**86;00**	81;50,24	0;00,45,52
64;00	64;35,25	0;00,53,13	**86;30**	82;13,19	0;00,45,40
64;30	64;02,02	0;00,53,4	**87;00**	82;36,09	0;00,45,29
65;00	64;28,34	0;00,52,55	**87;30**	82;58,54	0;00,45,18
65;30	64;55,01	0;00,52,46	**88;00**	83;21,33	0;00,45,06
66;00	65;51,24	0;00,52,37	**88;30**	83;44,04	0;00,44,55
66;30	65;47,43	0;00,52,28	**89;00**	84;06,32	0;00,44,43
67;00	66;13,57	0;00,52,19	**89;30**	84;28,54	0;00,44,31
67;30	66;40,07	0;00,52,10	**90;00**	84;51,10	0;00,44,20

Arc	Chord	Sixtieths	Arc	Chord	Sixtieths
90;30	85;13,20	0;00,44,08	**113;00**	110;03,59	0;00,34,34
91;00	85;35,24	0;00,43,57	**113;30**	100;21,16	0;00,34,20
91;30	85;57,23	0;00,43,45	**114;00**	100;38,26	0;00,34,06
92;00	86;19,15	0;00,43,33	**114;30**	100;55,28	0;00,33,52
92;30	86;41,02	0;00,43,21	**115;00**	101;12,25	0;00,33,39
93;00	87;2,42	0;00,43,09	**115;30**	101;29,15	0;00,33,25
93;30	87;24,17	0;00,42,57	**116;00**	101;45,57	0;00,33,11
94;00	87;45,45	0;00,42,45	**116;30**	102;02,33	0;00,32,57
94;30	88;07,07	0;00,42,33	**117;00**	102;19,01	0;00,32,43
95;00	88;28,24	0;00,42,21	**117;30**	102;35,22	0;00,32,29
95;30	88;49,34	0;00,42,09	**118;00**	102;51,37	0;00,32,15
96;00	89;10,39	0;00,41,57	**118;30**	103;07,44	0;00,32,00
96;30	89;31,37	0;00,41,45	**119;00**	103;23,44	0;00,31,46
97;00	89;52,29	0;00,41,33	**119;30**	103;39,37	0;00,31,32
97;30	90;13,15	0;00,41,21	**120;00**	103;55,23	0;00,31,18
98;00	90;33,55	0;00,41,08	**120;30**	104;11,02	0;00,31,04
98;30	90;54,29	0;00,40,55	**121;00**	104;26,34	0;00,30,49
99;00	91;14,56	0;00,40,42	**121;30**	104;41,59	0;00,30,35
99;30	91;35,17	0;00,40,30	**122;00**	104;57,16	0;00,30,21
100;00	91;55,32	0;00,40,17	**122;30**	105;12,26	0;00,30,07
100;30	92;15,40	0;00,40,04	**123;00**	105;27,30	0;00,29,52
101;00	92;35,42	0;00,39,52	**123;30**	105;42,26	0;00,29,37
101;30	92;55,38	0;00,39,39	**124;00**	105;57,14	0;00,29,23
102;00	93;15,27	0;00,39,26	**124;30**	106;11,55	0;00,29,08
102;30	93;35,11	0;00,39,13	**125;00**	106;26,29	0;00,28,54
103;00	93;54,47	0;00,39,00	**125;30**	106;40,56	0;00,28,39
103;30	94;14,17	0;00,38,47	**126;00**	106;55,15	0;00,28,24
104;00	94;33,41	0;00,38,34	**126;30**	107;09,27	0;00,28,10
104;30	94;52,58	0;00,38,21	**127;00**	107;23,32	0;00,27,56
105;00	95;12,09	0;00,38,08	**127;30**	107;37,30	0;00,27,40
105;30	95;31,13	0;00,37,55	**128;00**	107;51,20	0;00,27,25
106;00	95;50,11	0;00,37,42	**128;30**	108;05,02	0;00,27,10
106;30	96;09,02	0;00,37,29	**129;00**	108;18,37	0;00,26,56
107;00	96;27,47	0;00,37,16	**129;30**	108;32,05	0;00,26,41
107;30	96;46,24	0;00,37,03	**130;00**	108;45,25	0;00,26,26
108;00	97;04,55	0;00,36,50	**130;30**	108;58,38	0;00,26,11
108;30	97;23,20	0;00,36,36	**131;00**	109;11,44	0;00,25,56
109;00	97;41,38	0;00,36,23	**131;30**	109;24,42	0;00,25,41
109;30	97;59,49	0;00,36,09	**132;00**	109;37,32	0;00,25,26
110;00	98;17,54	0;00,35,56	**132;30**	109;50,15	0;00,25,11
110;30	98;35,52	0;00,35,42	**133;00**	110;02,50	0;00,24,56
111;00	98;53,43	0;00,35,29	**133;30**	110;15,18	0;00,24,41
111;30	99;11,27	0;00,35,15	**134;00**	110;27,39	0;00,24,26
112;00	99;29,05	0;00,35,01	**134;30**	110;39,52	0;00,24,10
112;30	99;46,35	0;00,34,48	**135;00**	110;51,57	0;00,23,55

Arc	Chord	Sixtieths	Arc	Chord	Sixtieths
135;30	111;03,54	0;00,23,40	**158;00**	117;47,43	0;00,11,51
136;00	111;15,44	0;00,23,25	**158;30**	117;53,39	0;00,11,35
136;30	111;27,26	0;00,23,09	**159;00**	117;59,27	0;00,11,19
137;00	111;39,01	0;00,22,54	**159;30**	118;5,7	0;00,11,03
137;30	111;50,29	0;00,22,39	**160;00**	118;10,37	0;00,10,47
138;00	112;01,47	0;00,22,24	**160;30**	118;16,1	0;00,10,31
138;30	112;12,59	0;00,22,08	**161;00**	118;21,16	0;00,10,14
139;00	112;24,03	0;00,21,53	**161;30**	118;26,23	0;00,09,58
139;30	112;35,00	0;00,21,37	**162;00**	118;31,22	0;00,09,42
140;00	112;45,48	0;00,21,22	**162;30**	118;36,13	0;00,09,25
140;30	112;56,29	0;00,21,07	**163;00**	118;40,55	0;00,09,09
141;00	113;07,02	0;00,20,51	**163;30**	118;45,30	0;00,08,53
141;30	113;17,25	0;00,20,36	**164;00**	118;49,56	0;00,08,37
142;00	113;27,44	0;00,20,20	**164;30**	118;54,15	0;00,08,20
142;30	113;37,54	0;00,20,04	**165;00**	118;58,25	0;00,08,04
143;00	113;47,56	0;00,19,49	**165;30**	119;2,26	0;00,07,48
143;30	113;57,50	0;00,19,33	**166;00**	119;6,20	0;00,07,31
144;00	114;07,37	0;00,19,17	**166;30**	119;10,6	0;00,07,15
144;30	114;17,15	0;00,19,02	**167;00**	119;13,44	0;00,06,59
145;00	114;26,46	0;00,18,46	**167;30**	119;17,13	0;00,06,42
145;30	114;36,09	0;00,18,30	**168;00**	119;20,34	0;00,06,26
146;00	114;45,24	0;00,18,14	**168;30**	119;23,47	0;00,06,10
146;30	114;54,31	0;00,17,59	**169;00**	119;26,52	0;00,05,53
147;00	115;03,30	0;00,17,43	**169;30**	119;29,49	0;00,05,37
147;30	115;12,22	0;00,17,27	**170;00**	119;32,37	0;00,05,20
148;00	115;21,06	0;00,17,11	**170;30**	119;35,17	0;00,05,04
148;30	115;29,41	0;00,16,55	**171;00**	119;37,49	0;00,04,48
149;00	115;38,09	0;00,16,40	**171;30**	119;40,13	0;00,04,31
149;30	115;46,29	0;00,16,24	**172;00**	119;42,28	0;00,04,14
150;00	115;54,40	0;00,16,08	**172;30**	119;44,35	0;00,03,58
150;30	116;02,44	0;00,15,52	**173;00**	119;46,35	0;00,03,42
151;00	116;10,40	0;00,15,36	**173;30**	119;48,26	0;00,03,26
151;30	116;18,28	0;00,15,20	**174;00**	119;50,8	0;00,03,09
152;00	116;26,08	0;00,15,04	**174;30**	119;51,43	0;00,02,53
152;30	116;33,40	0;00,14,48	**175;00**	119;53,10	0;00,02,36
153;00	116;41,04	0;00,14,32	**175;30**	119;54,27	0;00,02,20
153;30	116;48,20	0;00,14,16	**176;00**	119;55,38	0;00,02,03
154;00	116;55,28	0;00,14,00	**176;30**	119;56,39	0;00,01,47
154;30	117;02,28	0;00,13,44	**177;00**	119;57,32	0;00,01,30
155;00	117;09,20	0;00,13,28	**177;30**	119;58,18	0;00,01,14
155;30	117;16,04	0;00,13,12	**178;00**	11958,55	0;00,00,57
156;00	117;22,40	0;00,12,56	**178;30**	119;59,24	0;00,00,41
156;30	117;29,08	0;00,12,40	**179;00**	119;59,44	0;00,00,25
157;00	117;35,28	0;00,12,24	**179;30**	119;59,56	0;00,00,09
157;30	117;41,40	0;00,12,07	**180;00**	120;0,0	0;00,00,00

Practice Problem Solutions

Converting Sexagesimal to Decimal

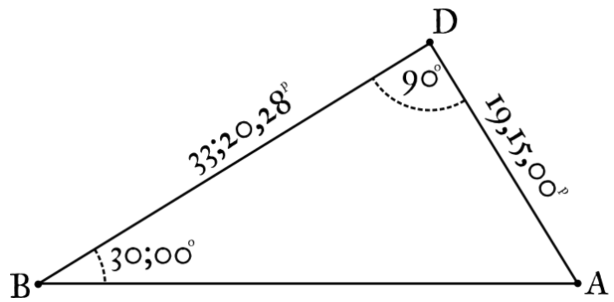
1. 196.485833…
2. 78.931388…
3. 63.004166…

Converting Decimal to Sexagesimal

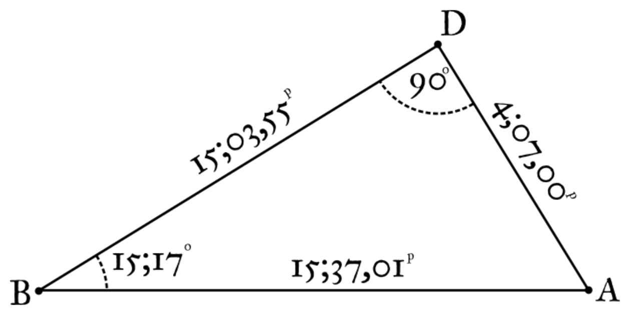
1. 3,20;18,44
2. 5,51;26,15
3. 1,33;51,23

Demi-Degrees

1.



2.



Menelaus' Equation

1. First form of the equation. $arc\ EM\ =\ 8;38°$.
2. Second form of the equation. $arc\ \Theta H\ =\ 32;10°$.